UDC 577.31

OPTIMAL STRUCTURES OF BRANCHING PIPELINES

PMM Vol. 41, № 2, 1977, pp. 376-383
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(Received April 16, 1976)

Branching pipelines through which liquid or gas enters some medium or is withdrawn from it are wide-spread in nature and technology (blood circulation and respiratory systems, irrigation systems, etc.). In the present work, simple hypotheses of similarity, of boundedness of the volume of transported fluid and of minimum hydraulic resistance, the pipeline configuration is determined, and basic principles are established for determination of the length and cross-sectional areas of pipelines in dependence of their order (distance from source). Results are compared with experimental data.

1. Statement of the problem. Let us consider a branching pipeline for the supply (or extraction) of liquid or gas to a certain region D_0 of a two- or three-dimensional space. The dimensionality of that space will be denoted by $\nu = 2, 3$. The pipeline is to begin at some point (source or sink) and the ends of its branches must cover the specified region D_0 fairly uniformly and densely to ensure the supply of transported fluid to the neighborhood of any point of the region.

We admit the following basic hypotheses.

1. The pipeline is laid out according to the hierarchical principle: each pipe of the n-th order branches into two pipes of the $(n + 1)$-th order, $n = 0, 2, \ldots, N$. In the terminology of the theory of graph the pipeline is a dichotomic tree.

2. All 2^n pipes of n-th order are of the same length l_n and of the same cross-sectional area s_n, $n = 0, 1, \ldots, N$.

3. Region D_n supplied by any n-th order pipe with all its branches is divided at the branching of that pipe into two symmetric regions D_{n+1} which are fed by $(n + 1)$-st order pipes. These regions are similar to region D_n. The branching point lies in the plane of symmetry of region D_n.

4. The optimality of the sought pipeline is defined as follows: its total hydraulic resistence is the lowest of all pipelines which satisfy hypotheses 1—3 and have the specified total volume Ω.

Note that the fulfilment of the 3-rd hypothesis ensures for a reasonably high N the supply of the transported fluid to any specified arbitrarily small neighborhood of any point of region D_0.

The 4-th hypothesis directly implies that all pipes must be straight. It is obvious that for any specified position of branching point it is the straight pipes that have the lowest hydraulic resistance for the specified volume.

Let us pass to the determination of the pipeline layout that would satisfy the imposed conditions. We have to determine the pipeline configuration, the shape of regions D_n supplied by n-th order pipes, the pipe lengths l_n, and their cross-sectional areas s_n.

2. The geometry of regions. We use the 3-rd hypothesis for determining the shape of regions D_n. Let us show that in the two-dimensional case regions D_n can be either rectangles with the ratio of sides

$$L_0 : L_1 = \sqrt{2} \qquad (2.1)$$

or isosceles right triangles. In the three-dimensional case regions D_n that satisfy the 3-rd condition are right angle parallelepipeds with the ratio of edges

$$L_0 : L_1 : L_2 = 2^{2/3} : 2^{1/3} : 1 \qquad (2.2)$$

First, we consider the two-dimensional case $\nu = 2$ and assume that the parent region D_0 is bounded by a piecewise smooth curve consisting of segments of straight lines and curved arcs. Let the over-all length of curvilinear section arcs be a. As the result of branching region D_0 is divided into 2^n equal regions D_n similar to region D_0. All linear dimensions of region D_n are $2^{n/2}$ -times smaller than those of region D_0, hence the over-all length of curvilinear sections of a single D_n region is equal $2^{-n/2}a$, and for all 2^n regions D_n it is equal $2^{n/2}a$. However each region is divided in half by its axis of symmetry at bifurcation, which means that no new curvilinear sections are added. The over-all length of all curvilinear sections of boundaries of D_n regions is thus equal a and simultaneously equal $2^{n/2}a$. This implies that $a = 0$ and that all D_n regions are polygons, $n = 0, 1, \ldots, N$. It can be similarly proved that in the three-dimensional case all D_n regions are polyhedrons.

Let us determine the possible shapes of polygons D_0 (or D_n) in the two-dimensional case. The polygon D_0 must have an axis of symmetry which divides it into two equal polygons D_1 similar to D_0. It follows from this that D_0 and D_1 are simply connected polygons. Let k be the number of sides of polygon D_0 and let its axis of symmetry intersect k_1 of these. Owing to simple connectedness, $k_1 \leqslant 2$. The over-all number of sides of polygons D_1 is $2k$. On the other hand, that number is $k + k_1 + 2$, since at each bifurcation each of the k_1 sides is divided in two, and besides this, a side is added to each polygon owing to symmetry. Equality $2k = k + k_1 + 2$ and the inequality $k_1 \leqslant 2$ imply that $k \leqslant 4$, i.e. that D_0 is either a quadrangle or a triangle.

When D_0 is a quadrangle, the axis of symmetry must divide it in two quadrangles. Hence the axis of symmetry intersects two opposite sides of the quadrangle and owing to symmetry it must be perpendicular to these sides. Consequently these sides are parallel and region D_0 is a trapezoid and, because of symmetry, it must be isosceles. The regions D_1 into which it is divided by the axis of symmetry are rectangular trapezoids similar to D_0. This shows that D_0 is an isosceles rectangular trapezoid, i.e. a rectangle. We denote the larger and the smaller sides of this rectangle D_0 by L_0 and L_1, respectively. If the rectangle with obtained by halving D_0 is to be similar to the latter, its sides L_1 and $L_0 / 2$ must satisfy the relationship (2.1).

When D_0 is a triangle, it must have an axis of symmetry, i.e. be an isosceles triangle. Its division yields a right triangle, hence, because of similarity, the parent triangle must also be a right one.

Thus in the two-dimensional case region D_0 and all D_n regions can be either rectangles with sides whose ratio satisfies (2.1), or isosceles right triangles.

Let us now consider the three-dimensional case. Let D_0 be a polyhedron of k faces with the plane of symmetry intersecting k_1 of these. A reasoning similar to that in the two-dimensional case yields the equality $k = k_1 + 2$. Hence only two faces of polyhedron D_0 are not intersected by the axis of symmetry. These faces are symmetric with respect to the plane of symmetry; we shall call them "bases". The remaining k_1 faces are normal to the plane of symmetry and will be called "lateral" faces. Consequently, the polyhedron D_0 is the result of truncation of k_1-faced prism by two symmetric bases. The lateral faces are symmetric with respect to the plane of symmetry of region D_0 and can be isosceles trapezoids or isosceles triangles. The bases are polygons of j sides with $j = k_1$ when the base has no common points with plane of symmetry or has a single vertex in that plane, and $j = k_1 + 1$ when one side of the base lies in the plane of symmetry. The halving of region D_0 yields two polyhedrons, and in the plane of symmetry the faces have j sides. Because of the similarity of obtained polyhedrons to the parent one, all their dimensions are $2^{1/3}$ times smaller and the areas of similar faces $2^{2/3}$ times smaller than the corresponding dimensions and areas of D_0.

There are three mutually exclusive cases:

a) all lateral faces are triangles and $j = 3$;

b) all lateral faces are trapezoids and $j = 4$;

c) there is at least one lateral face (a triangle or a trapezoid) the number of whose sides is not equal to the number of sides of the base j.

In the case (c) the over-all area of all those faces whose number of sides is not equal j, is reduced by half when the polyhedron D_0 is divided in half. Hence in that case the condition of area similarity is not satisfied.

In case (a) region D_0 is a triangular pyramid $ABCD$ whose plane of symmetry is BCK (Fig. 1). Angles AKC and AKB are right angles, since the plane of symmetry is perpendicular to edge AD. Thus two right angles adjoin the vertex K of pyramid $ABCK$. Since pyramid $ABCK$ is similar to the parent pyramid $ABCD$, hence two right angles must adjoin one of the vertices of the latter. Owing to symmetry triangles ACD and ABD are isosceles and their angles CAD, ADC, BAD and ADB are acute. Hence two right angles in pyramid $ABCD$ can only adjoin vertices B or C. Let the two right angles adjoin vertex B. Because of symmetry these must be angles ABC and DBC. Thus in the pyramid $ABCK$ the face ABC which does not adjoin vertex K is a right triangle. In the parent pyramid $ABCD$ vertex B is the analog of vertex K in pyramid $ABCK$ to which adjoin two right angles. Hence face ACD which does not adjoin it must also be a right triangle.

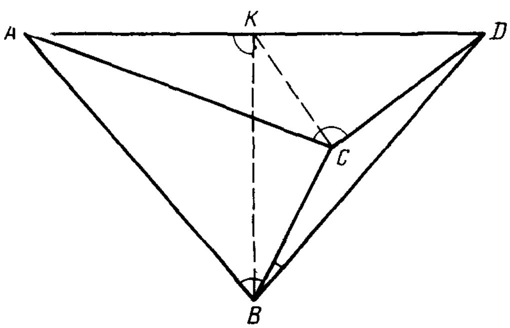

Fig. 1

Thus triangles ACD and ABC are isosceles and right, and face ABC of pyramid $ABCK$ is the analog of face ACD of the parent pyramid $ABCD$ (both these faces are opposite vertices K and B, respectively, of these pyramids). Consequently the linear dimensions of these triangles must differ by a factor of $2^{1/3}$. However the leg AC of triangle ACD is the hypotenuse of triangle ABC, hence their linear dimensions differ by a factor of $2^{1/2}$. This contradiction eliminates case (a).

In the remaining case (b) the polyhedron D_0 has only six tetragonal faces of which four are lateral faces and two are bases. The plane of symmetry is perpendicular to the four lateral faces, hence after division the polyhedron obtains a face that is perpendicular to four adjacent faces. Hence the parent polyhedron must also have a face that is normal to all four faces adjacent to it. Whether that face is a lateral or a base face, the base is in both cases perpendicular to lateral faces. This means that D_0 is a straight prism with a tetragonal base. If the base is not a tetragon, the halving reduces the overall area of all rectangular (lateral) faces by half, which contradicts the condition of area reduction by a factor of $2^{2/3}$ at division. Hence all faces of D_0 are rectangles and D_0 is a rectangular parallelepiped. We denote its edges by L_0, L_1 and L_2 with $L_0 \geqslant L_1 \geqslant L_2$. The plane of symmetry which is normal to the longest edge must divide parallelepiped D_0 into two parallelepipeds similar to the parent one. Hence we have

$$L_0 : L_1 : L_2 = L_1 : L_2 : (L_0 / 2)$$

Formula (2.2) follows from this ratio. Thus in the three-dimensional case the rectangular parallelepiped whose edges satisfy the relation (2.2) is the only shape that satisfies the 3-rd hypothesis.

3. The length of pipes. Let us derive the expression for the length l_n of n-th order pipes. Let L_n be the characteristic linear dimension of region D_n equal to its longest side with the ratio of sides satisfying (2.1) in the case of rectangular regions; in the case of isosceles right triangle regions, equal to the foot of the triangle, and in the case of parallelepipeds whose edges satisfy the relation (2.2) to the longest edge. Since each division reduces all linear dimensions by a factor of $2^{1/\nu}$, hence

$$l_n = L_n\varphi_n, \quad L_n = 2^{-n/\nu}L_0; \quad \nu = 2, 3; \quad n = 0, 1, \ldots, N \tag{3.1}$$

The dimensionless quantity φ_n depends on the space dimensionality ν, the shape of the region, and on the position of branching points which lie in the corresponding planes of symmetry.

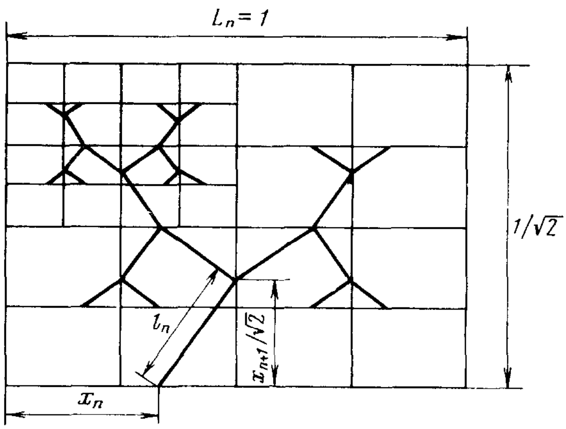

Fig. 2

In the case of rectangular region D_n whose sides satisfy relation (2.1), we denote by x_n the ratio in which the longest side L_n of the rectangle D_n is divided by the beginning of the n-th order pipe. Elementary geometrical considerations (see Fig. 2 where $L_n = 1$) show that

$$\varphi_n = \varphi(x_n, x_{n+1}) = [(x_n - {}^1/_2)^2 + x_{n+1}^2 / 2]^{1/2} \qquad (3.2)$$
$$0 \leqslant x_n \leqslant 1, \quad n = 0, 1, \ldots, N$$

For region D_n in the form of an isosceles right triangle x_n represents the ratio in which the leg of the triangle is divided by the beginning of the n-th order pipe. The quantity x_n is read off from the right angle vertex of the triangle (see Fig. 3 where $L_n = 1$). We have

$$\varphi_n = \varphi(x_n, x_{n+1}) = \{[x_n - (1 - x_{n+1}) / 2]^2 + [(1 - x_{n+1}) / 2]^2\}^{1/2} \qquad (3.3)$$

In the three-dimensional case of the parallelepiped whose ratio of ribs satisfies (2.2) we set, for simplicity, $L_n = 1$. We set a Cartesian system of coordinates so that it coincides with the parallelepiped axes and that the parallelepiped lies in the first octant

$$0 \leqslant x \leqslant 1, \quad 0 \leqslant y \leqslant 2^{-1/3}, \quad 0 \leqslant z \leqslant 2^{-2/3}$$

Let the beginning of the n-th order pipe lie in the plane $z = 0$ at coordinates x_n, y_n related to the dimension L_n. The end of that pipe at which it branches into pipes of order $(n + 1)$ lies in the plane of symmetry $x = {}^1/_2$ of the parallelepiped D_n at coordinates ${}^1/_2$, $x_{n+1}2^{-1/3}$, $y_{n+1}2^{-1/3}$. We take here into account that $L_n = 1$ and $L_{n+1} = 2^{-1/3}$. From this we obtain for φ_n from formula (3.1) an expression of the form

$$\varphi_n = \varphi(x_n, y_n; x_{n+1}, y_{n+1}) = [(x_n - {}^1/_2)^2 + (y_n - x_{n+1}2^{-1/3})^2 + y_{n+1}^2 2^{-2/3}]^{1/2}, \quad 0 \leqslant x_n \leqslant 1, \quad 0 \leqslant y_n \leqslant 2^{-1/3}, \quad n = 0, 1, \ldots, N \qquad (3.4)$$

If one takes into account that not only regions D_n are similar to each other for various n, but also the points of branching have similar positions relative to corresponding regions, it is necessary to set in formulas (3.2) and (3.3)

$$x_n = x^*, \quad \varphi_n = \varphi(x^*, x^*) = \varphi^* \qquad (3.5)$$

in formula (3.4) we then have

$$x_n = x^*, \; y_n = y^*, \quad \varphi_n = \varphi(x^*, x^*; y^*, y^*) = \varphi^* \qquad (3.6)$$

where x^* and y^* are some constants. We shall call such pipelines regular.

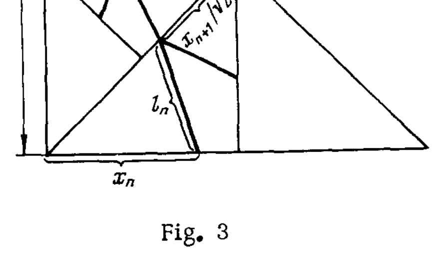

Fig. 3

The values of x^* and y^* for regular pipelines at which parameter φ^* in (3.5) or (3.6) reaches its maximum are of interest. They correspond to the optimal regular configuration of the pipeline with the minimum length of all pipes. To determine such configuration in the case of rectangular regions we substitute $x_n = x_{n+1} = x^*$ into formula (3.2) and find its minimum with respect to x^*. As the result we have

$$x^* = {}^1/_3, \quad \varphi^* = 12^{-1/2} \approx 0.2887 \qquad (3.7)$$

Similar calculations by formula (3.3) for triangular regions yield

$$x^* = 0.4, \quad \varphi^* = 10^{-1/2} \approx 0{,}3162 \qquad (3.8)$$

For three-dimensional regions in the form of parallelepipeds the determination of minimum of function (3.4) with respect to variables $x_n = x_{n+1} = x^*$ and $y_n = y_{n+1} = y^*$, yields the following relationships:

$$x^* = 2^{-1/3}(2^{4/3} - 1)\,3^{-1} \approx 0.4002 \qquad (3.9)$$
$$y^* = (2^{2/3} - 1)\,3^{-1} \approx 0.1958$$
$$\varphi^* = (2^{2/3} - 1)^{1/2}2^{-1}3^{-1/2} \approx 0.2212$$

Let us compare the optimal regular pipelines for rectangular and triangular plane regions as to the length of pipes per unit area. Because of similarity it is sufficient to compare the length l_n of n-th order pipes for regions D_n of one and the same area σ. For a rectangular region D_n whose sides conform to (2.1) and the maximum of these is L_n, and for an isosceles right triangle with legs of length L_n we have, respectively,

$$L_n^2 2^{-1/2} = \sigma, \quad L_n^2 / 2 = \sigma \qquad (3.10)$$

We determine L_n in formulas (3.10) in terms of σ, substitute the derived expressions and also equalities (3.7) and (3.8) for φ^* into formulas (3.1) for l_n, and obtain

$$l_n = \sigma^{1/2}2^{-3/4}3^{-1/2} \approx 0.3423\,\sigma^{1/2} \qquad (3.11)$$
$$l_n = \sigma^{1/2}5^{-1/2} \approx 0.4472\sigma^{1/2}$$

for rectangles and triangles, respectively. It is seen from (3.11) that the optimal regular pipeline with rectangular regions has a smaller length of pipes per unit area than a similar pipeline with triangular regions.

4. **Minimization of resistance.** With allowance for formula (3.1) the overall volume of the pipeline is

$$\Omega = \sum_{n=0}^{N} 2^n l_n s_n = L_0 \sum_{n=0}^{N} 2^{-n-n/\nu}\varphi_n s_n \qquad (4.1)$$

Let p_n be the pressure at the beginning of the n-th order pipe and Q be the total fluid flow rate. We assume that the pressure drop in a single pipe is defined by the Poiseuille law for the laminar flow of a viscous incompressible fluid [1]

$$p_n - p_{n+1} = c\mu q_n l_n s_n^{-r}, \quad q_n = 2^{-n}Q \qquad (4.2)$$

where μ is the viscosity of the fluid, q_n is the rate of flow of the fluid through a single n-th order pipe, and the coefficient c depends on the cross section shape. For pipes of similar cross sections (e.g. round) the coefficient r is equal 2. For channels or pipes of rectangular cross sections of various widths and constant depth $r = 3$. The ratio of overall pressure drop $p_0 - p_{N+1}$ to the flow rate Q through the pipeline in conformity with (4.2) and (3.1) is defined by

$$R = \sum_{n=0}^{N} 2^{-n} l_n s_n^{-r} = L_0 \sum_{n=0}^{N} 2^{-n-n/\nu}\varphi_n s_n^{-r} \qquad (4.3)$$

to within the constant factor.

Let us first consider the limit case of $N \to \infty$. We assume that φ_n are bounded (e.g. are equal to the constant φ^*). For series (4.1) and (4.3) to be convergent it is necessary that the n-th terms of these tend to zero. From this we obtain the conditions

$$s_n = \alpha_n 2^{(1/\nu-1)n} = \beta_n^{-1}2^{-(1/\nu+1)n/r}, \ \alpha_n \to 0, \ \beta_n \to 0, \ n \to \infty \qquad (4.4)$$

where α_n and β_n are infinitely small. Conditions (4.4) are not incompatible when the inequality

$$r < (\nu + 1) / (\nu - 1) \tag{4.5}$$

is satisfied. For a pipeline consisting of channels of constant depth and various widths ($r = 3$) which feeds the two-dimensional region ($\nu = 2$), and also for a pipeline of round pipes ($r = 2$) in the three dimensional case, inequality (4.5) is not satisfied and is converted to an equality. Thus the hydraulic resistance R of such pipelines is unbounded when volume Ω is finite and $N \to \infty$. When $\nu = 3$ and $r = 3$, inequality (4.5) is also violated. It is satisfied only for round pipes (or generally for pipes of similar cross sections) that feed a plane region ($r = \nu = 2$). Only in such case an infinitely branching pipeline of limited volume and finite hydraulic resistance is feasible. If we assume

$$s_n = s_0 2^{-\gamma n}; \quad n = 0, 1, 2, \ldots; \quad {}^1/_2 < \gamma < {}^3/_4, \quad r = \nu = 2 \tag{4.6}$$

then both series (4.1) and (4.3) become convergent.

In the case of finite N we determine cross sections s_n by the 4-th hypothesis that the hydraulic resistance R is minimum with respect to s_n when restrictions (4.1) on the volume are satisfied. Using the method of Lagrange multipliers and calculating the minimum of R with respect to s_n under condition (4.1), we obtain

$$s_n = s_0 2^{-2n/(r+1)}, \quad n = 0, 1, \ldots \tag{4.7}$$

Note that the relationships (4.7) are independent of φ_n. Substituting expressions (4.7) into the sums (4.1) and (4.3), we obtain

$$\Omega = L_0 s_0 F, \quad F = \sum_{n=0}^{N} 2^{\delta n} \varphi_n, \quad \delta = \frac{r-1}{r+1} - \frac{1}{\nu} \tag{4.8}$$

$$s_0 = \Omega (L_0 F)^{-1}, \quad R = L_0 s_0^{-r} F = L_0^{r+1} \Omega^{-r} F^{r+1}$$

Formulas (4.7) and (4.8) determine the cross-sectional areas s_n including s_0 and the hydraulic resistance R of the pipeline for specified φ_n Substituting the expressions (3.2) and (3.3) for φ_n in formula (4.8) for F, we have in the plane case

$$F = \sum_{n=0}^{N} 2^{\delta n} \varphi(x_n, x_{n+1}) \tag{4.9}$$

Various problems can be formulated and solved by minimizing the sum (4.9) for F with respect to x_n for various restrictions on the position of branching points x_n. The most general restriction is of the form $0 \leqslant x_n \leqslant 1$, where $n = 0, 1, \ldots, N$. The minimum of F in (4.9) corresponds according to (4.8) to the minimum of hydraulic resistance R. The three-dimensional case can be similarly considered with the use of formula (3.4) for φ_n.

Let us consider regular pipelines which satisfy relationships (3.5) and (3.6). Substituting (3.5) and (3.6) into equalities (4.8) we obtain the simple formulas

$$F = \varphi^* [2^{(N+1)\delta} - 1] (2^\delta - 1)^{-1}, \quad \delta \neq 0 \tag{4.10}$$

$$F = (N + 1) \varphi^*, \quad \delta = 0 \quad (\nu = 3, r = 2; \quad \nu = 2, r = 3)$$

The second of formulas (4.10) is valid for pipes in the three-dimensional case and for channels in the two-dimensional one.

Parameters φ^* for the optimum regular pipeline are determined by equalities (3.7)—(3.9) for the corresponding configurations. Thus in this case all pipeline parameters are specified by formulas (3.1),(3.5)—(3.9),(4.7),(4.8) and (4.10). Such pipleline for rectangular regions is shown in Fig. 2, where the source lies on the longer side of the rectangle and divides it in the ratio 1 : 3. If we omit in Fig. 2 pipe l_n and take its branching point as the source, we obtain an optimal regular pipeline with its source at the axis of symmetry. The position point of the source and all branching points divide corresponding sides of rectangles in the ratio 1:3. Note that the rejection of conditions (3.5) and (3.6), i.e. of the condition of regularity, makes it possible to obtain by minimizing the sum (4.9) a lower hydraulic resistance than that of the optimal regular pipeline.

5. Generalization and comparison with experiment. The pipelines considered above possess the property of similarity (the 3-rd hypothesis) and regular pipelines have, in addition, a similar disposition of pipeline branches with respect to these regions. As shown in Sect. 2, these properties can be strictly achieved only for some particular shapes of regions, while for regions of another shape this is not feasible. However the determined here configurations appear to have some asymptotic meaning for regions of an arbitrary shape. Since after numerous branchings the dependence of the pipeline configuration on the shape of the parent region shape levels out — an assumption that appears natural — the configuration approaches the derived above.

Restriction (4.1) on the pipeline volume Ω may be replaced by one imposed on the over-all quantity of material or cost of the pipeline. If the thickness of pipe walls is assumed proportional to the cross-sectional area of these, the restriction on the quantity of material is of the form (4.1).

In actual system a strict hierarchy of branches is not usually satisfied, hence branches of various orders are conventionally combined on a certain principle into one order. This means that the pipe of each order is divided in m branches, where m is generally a fractional number, and not in two. The regions are in that case not similar, it is, however, possible to obtain similar relationships for the lengths and cross sections, which are valid in the average. Since at branching of pipes related areas decrease in the average by a factor of m, hence for the length of pipes we obtain similarly to (3.1)

$$l_n = l_0 m^{-n/\nu}, \quad n = 0, 1, \ldots, N \tag{5.1}$$

Formulas for volume Ω and the hydraulic resistance R are of a form similar to formulas (4.1) and (4.3)

$$\Omega = l_0 \sum_{n=0}^{N} m^{-n-n/\nu} s_n, \quad R = l_0 \sum_{n=0}^{N} m^{-n-n\nu} s_n^{-r} \tag{5.2}$$

The condition of minimum R with respect to s_n for the fixed volume defined by (5.2) yields formulas

$$s_n = s_0 m^{-2n/(r+1)}, \quad s_0 = \Omega l_0^{-1} F_1^{-1}, \quad n = 1, 2, \ldots, N \tag{5.3}$$

$$R = l_0 s_0^{-r} F_1 = l_0^{r+1} \Omega^{-r} F_1^{r+1}$$

$$F_1 = \sum_{n=0}^{N} m^{\delta n} = [m^{\delta(N+1)} - 1](m^{\delta} - 1), \quad \delta \neq 0$$

$$F_1 = N + 1, \quad \delta = 0$$

where δ is determined by formula (4.8), which are similar to (4.7)—(4.9).

Let us compare the obtained relationships with experimental data on branching (human arteries and lungs) presented in [2]. It is shown there on the basis of a large number of observations that for the considered arteries

$$m = 3.096, \quad \lg l_n = -0.172 \ n + \text{const} \qquad (5.4)$$
$$\lg d_n = -0.2015 \ n + \text{const}$$

where the usual notation is used and d_n is the diameter of the artery. The substitution of the expression (5.4) for m into formulas (5.1) and (5.3), and also $\nu = 3$ and $r = 2$, yields

$$\lg l_n = -(n \mathbin{/} 3) \lg m + \text{const} = -0.1636 \ n + \text{const} \qquad (5.5)$$
$$\lg d_n = (^1/_2) \lg s_n + \text{const} = -(n \mathbin{/} 3) \lg m + \text{const} = -0.1636 \ n + \text{const}$$

The difference in the coefficients of formulas (5.4) and (5,5) is 5% for lengths and 23% for the diameters of vessels.

Note that the geometry of branching pipelines under the condition of equality of angles between each pipe and its branches was considered in [3] and in the recently published paper [4]. We note that under such condition the requirement for the similarity of regions fed by pipes of various orders specified in this paper is not satisfied.

The author thanks V. M. Entov for his interest in this work, and V. M. Khaiutin and A. M. Mel'kumiants for the valuable discussion of physiological aspects.

REFERENCES

1. Kochin, N. E., Kibel', I. A. and Roze, N. V., Theoretical Hydrodynamics, pt. 2. Moscow, Fizmatgiz, 1963.

2. Singhal, S., Henderson, R., Horsfield, K., Harding, K. and Cumming, G., Morphometry of the human pulmonary arterial tree. Circulation Res., Vol. 33, № 2, 1973.

3. Horsfield, K. and Cumming, G., Angles of branching and diameters of branches in the n → h human bronchial tree. Bull. Biophys., Vol. 20, 1967.

4. Warner, W. H. and Wilson, T. A., Distribution of end-points of a branching network with decaying branch length. Bull. Math. Biol., Vol. 38, № 3, 1976.

Translated by J. J. D.